Outerplanar Graphs with Proper Touching Triangle Representations

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Abstract. A touching triangle representation of a planar graphs consists of triangles representing vertices with pairs of adjacent triangles with non-empty common boundaries representing the edges. We study the problem of recognizing planar graphs with *proper touching triangle representation*, where the union of all triangles is itself a triangle without holes. It has been conjectured that testing whether a planar graph is a proper touching triangle graph (TTG) can be done in polynomial time. Here we provide a necessary condition for a biconnected outerplanar graph to be a proper TTG and provide a slightly weaker sufficient condition. Together these two also give a characterization for a more restricted class of outerplanar graphs.

1 Introduction

While the node-link representation is the most popular way of drawing planar graphs, many others representations have been considered. Contact representations have been studies as far back as Koebe's 1936 "kissing circles" representation [11]. Since then many other variants have been considered including triangle contact representations [5, 9], and even cube contact representations in 3D [14, 7]. Here we study contact representations of planar graphs, where vertices are represented by simple polygons and edges are represented by non-trivial contact between the sides of two polygons. For practical, cognitive and aesthetic reasons, it is desirable to limit the polygonal complexity (as measured by the number of sides of the polygons) and the unused area in the representation (also known as holes). It is known that convex hexagons are always sufficient and sometimes necessary for such representations of planar graphs [3, 6]. A natural problem is to characterize classes of planar graphs that can be represented by polygons with fewer than six sides.

While there is no characterization of planar graphs that are representable by touching pentagons, Ueckerdt shows that Hamiltonicity of the planar graph is a sufficient condition [15]. The quadrilateral case is well-studied and there is a complete characterization of the class of planar graphs that can be represented by axis-aligned rectangles namely maximally planar graphs without filled triangles [4, 13, 12]. For the seemingly simplest case of planar graph representable by touching triangles, there is much less known. If one allows the outer-boundary of the representation to be of arbitrary complexity, then it is known that several classes of planar graphs (e.g., grid graphs, outerplanar graphs) have such touching triangle representations [8, 1]. There is also a characterization for a restricted formulation of the problem, where if two vertices are adjacent in the graph then the corresponding two triangles must share an entire side in the TTG representation [8].

However, the most natural version of the problem is the one where we ask for the class of graphs that have *proper touching triangle representation (*TTG*)*, where the union of all triangles is itself a triangle without holes. This is exactly the problem that we consider in this paper. In particular, we provide a necessary condition for a biconnected outerplanar graph to be a proper TTG. We also provide a slightly weaker sufficient condition. Together these also give a characterization for a more restricted class of outerplanar graphs. To the best of our knowledge, the only other results about proper TTGs are in [10], where a fixed-parameter tractable decision algorithm for 3-connected planar max-degree-Δ graphs is described, and where it it shown that planar 3-connected cubic graphs are proper TTGs.

2 Preliminaries

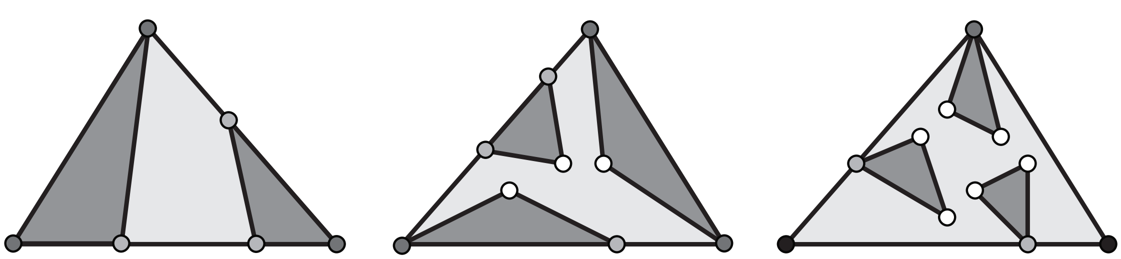
Let O be a biconnected outerplanar graph (BOPG) with an outerface f_o and a set F_I of internal faces, given by an outerplanar embedding of O. Let $deg_O(v)$ denote the degree of a vertex v in O. A *chain* in O is a path $v_1 v_2 \dots v_f$ of O where $deg_O(v_1) > 2$, $deg_O(v_f) > 2$ and each of the internal vertices v_2, $\dots$, v_{f-1} of the path has degree 2 in O. For any biconnected outerplanar graph O, it is always possible to iteratively delete a chain from the graph until it has only one edge. This iterative deletion of chains gives a *peeling order* of O. At each iteration this chain of vertices $v_1 v_2 \dots v_f$ along with the edge (v_1, v_f) forms an internal face that corresponds to a leaf in the weak dual. Thus the peeling order can also be thought of an ordering of the internal faces of O that iteratively constructs O. Formally, *a peeling order* of O is the bijection $f : \{1, \dots, k\} \xrightarrow[\text{onto}]{1-1} F_I$, where $k = |F_I|$ is the number of internal faces in O. Let O_i denote the subgraph of O induced by the faces $f(1)$, $\dots$, $f(i)$ for $i \in \{1, \dots, k\}$. Then the sequence of subgraphs O_1, O_2, $\dots$, $O_k = O$ is called the *subgraph-sequence of O induced by f*. A *chord* is an edge of O not on the outerface f_o. A *chord-only face* in O is a face that has no outer edge. The degree $deg_O(f)$ of a face f in O is the number of internal faces adjacent to f. Note that the degree of a face may be different than the number of edges on its boundary.

A *proper touching triangle representation* $\mathcal{R}_G$, or *proper* TTG of a planar graph $G = (V, E)$ is a set T of triangles with an isomorphism $\mathcal{T} : V \to T$ where the union of these triangles is a triangle and for any two vertices $u, v \in V$ the boundaries of $\mathcal{T}(u)$ and $\mathcal{T}(v)$ share a non-empty line-segment if and only if $(u, v) \in E$. For convenience, we often denote by $\triangle_u$ the triangle representing a vertex u of G in $\mathcal{R}_G$, i.e., $\triangle_u = \mathcal{T}(u)$.

We now define three types of triangles present within a proper TTG representation of a biconnected outerplanar graph; see Fig 1.

Definition 1. *Let $\mathcal{R}_O$ be a proper* TTG *representation for biconnected outerplanar graph O. A corner of a triangle in $\mathcal{R}_O$ is either an* exterior corner (X-corner) *when it is on the boundary of $\mathcal{R}_O$ or an* interior corner (I-corner)*, otherwise. An* X*-corner then is either an* apex exterior corner (A-corner) *when it is an apex of the boundary of $\mathcal{R}_O$ or a* non-apex exterior corner (B-corner)*, otherwise.*

(a) Corner triangles *have no* I*-corners and one or two* A*-corners; see Fig. 1(a).*
(b) Side triangles *have one* I*-corner and one or two* A*-corners; see Fig. 1(b).*
(c) Point triangles *have two* I*-corners and an* X*-corner; see Fig. 1(c).*



(a) corner triangles (b) side triangles (c) point triangles

Fig. 1: Types of triangles in a proper TTG representation.

Observation 2. *Let $O(V, E)$ be a* BOPG *with internal faces F_I and outerface f_o with $\mathcal{D}^*_O$ and $\mathcal{D}_O$ denoting the strong and weak duals, respectively, of O, and let $\mathcal{R}_O$ be its proper* TTG *representation.* X*-corners in $\mathcal{R}_O$ represent faces in the strong dual $\mathcal{D}^*_O$ incident to the vertex in $\mathcal{D}^*_O$ corresponding to f_o. Correspondingly,* I*-corners are the vertices of the weak dual $\mathcal{D}_O$, and hence, correspond distinctly to the $|F_I|$ internal faces of O. Thus, there are at most $|f_o|$* X*-corners and exactly $|F_I|$* I*-corners in $\mathcal{R}_O$.*

Proof. Consider the representation $\mathcal{R}_O$ of O, which in itself forms a graph $\mathcal{D}^*$ whose vertices are either X-corners or I-corners and whose edges are the sides of the triangles. Observe that the weak dual of $\mathcal{D}^*$ is O. Moreover, after contracting each edge of $\mathcal{D}^*$ connecting two X-corners (which corresponds to side of a triangles along the outerboundary of $\mathcal{R}_O$) yields $\mathcal{D}^*_O$, the strong dual of O.

Hence, X-corners correspond to faces in $\mathcal{D}^*_O$ incident to f^*_o, the vertex in $\mathcal{D}^*_O$ corresponding to f_o. Incident triangles of X-corners represent endpoints of one or more consecutive edges in f_o, which correspond distinctly to subpaths partitioning f_o. Thus, there are at most $|f_o|$ X-corners. Point, edge and corner triangles (triangles with an X-corner) represent the vertices in f_o. On the otherhand, I-corners are the vertices in $\mathcal{D}^*_O$ other than f^*_o, namely the vertices of the weak dual $\mathcal{D}_O$. Thus, the I-corners correspond distinctly to the $|F_I|$ internal faces of O. □

Next we define the notion of a "charge" of a vertex of a planar graph, which gives our first necessary condition for a planar graph to be a proper TTG.

Definition 3. *The* charge *of a vertex v in a planar graph G is*

$$ch(G, v) = \max\left\{deg_G(v) - 3,\ 0\right\}.$$

The total charge $ch(G)$ of G is the summation of the charges of all vertices of G. Each internal face can provide at most one charge to an incident vertex. A total charge function *$\Pi_G : F' \to V$ then allots a subset F' of internal faces to their incident vertices so that each vertex v is alotted at least $ch(G, v)$ faces.*

Lemma 4. *A planar graph is a proper* TTG only if *it has a total charge function.*

Proof. Let G be a planar graph with a proper TTG representation $\mathcal{R}_G$ as in Fig. 2. The degree of a vertex v in G corresponds to the number of triangles adjacent to the triangle $\triangle_v$ representing v in $\mathcal{R}$. Thus if we consider $\mathcal{R}$ as a graph, then the face $\triangle_v$ has exactly $deg(v)$ vertices. Each of these vertices is either an X-corner or an I-corner. Since the triangle $\triangle_v$ has exactly three apexes, $deg(v) - 3$ of these corners assume an $180°$ angle inside $\triangle_v$. However, an X-corner cannot assume an $180°$ angle inside $\triangle_v$; thus at

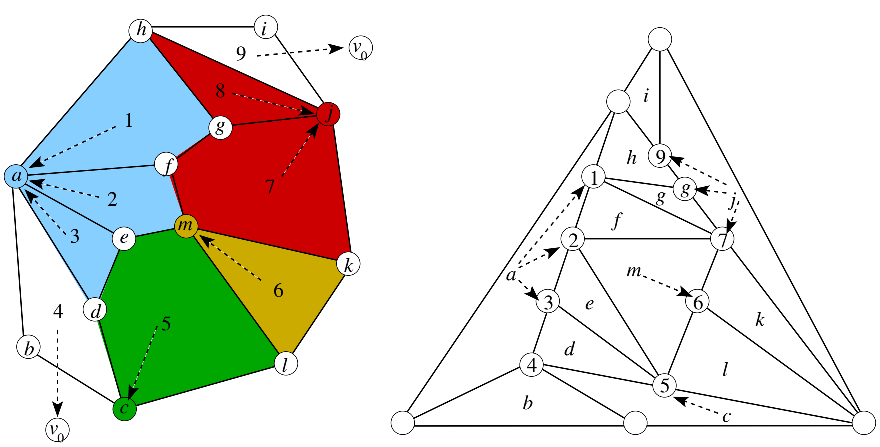


Fig. 2: Illustration for Lemma 4.

least $\deg(v) - 3$ I-corners assumes 180° angles inside $\triangle_v$. By Observation 2, these I-corners correspond to distinct internal faces of G. If we map these $deg(v) - 3$ internal faces to v, we find the desired total charge function since no I-corner can assume more than one 180° angles from different triangles. □

Observation 5. *For a planar graph G with total charge function Π_G, any face-induced subgraph H on the internal faces $F_H \subseteq F_I$, also has a total charge function Π_H, namely Π_G restricted to the faces of F_H.*

Proof. For each face $f_h \in F_H$, define $\Pi_H(f_h) = \Pi_G(f_h)$ so that $\Pi_H : F'_H \to V_H$ where $F'_H \subseteq F_H$.

Suppose that vertex v has x fewer incident faces in F_H than in F_G, where $|adj_v(F_H)| < |adj_v(F_G)|$ so that $x = |adj_v(F_G)| - |adj_v(F_H)|$. Then, v must also have x fewer incident edges so that $deg_H(v) = deg_G(v) - x$. Hence, the net charge of v in Π_H would then be

$$\begin{aligned} ch(\Pi_H, v) = ch(H, v) - |\Pi_H^{-1}(v)| &= (deg_H(v) - 3) - (|\Pi_G^{-1}(v)| \cap adj_v(F_H)|) \\ &= (deg_G(v) - x - 3) - (|\Pi_G^{-1}(v)| - x) \\ &= (deg_G(v) - 3) - |\Pi_G^{-1}(v)| = ch(G, v) - |\Pi_G^{-1}(v)| = ch(\Pi_G, v). \end{aligned}$$

Since, $ch(\Pi_G, v) = 0$ given that Π_G is a total charge function for G, then $ch(\Pi_H, v) = 0$ showing that Π_H is indeed a total charge function for H as claimed. □

Together with Lemma 4, Observation 5 gives the following restriction for a planar to be a proper TTG.

Corollary 6. *Let G be a planar graph G such that a face-induced subgraph of G is not a proper* TTG*. Then G is also not a proper* TTG.

3 Assigned Peeling Order and Proper TTG Representations

We saw in the previous section how we can construct a biconnected outerplanar graph starting from an edge and iteratively augmenting it with a chain. We want to use this peeling order of a biconnected outerplanar graph to obtain a proper TTG.

Lemma 7. *Let O be a biconnected outerplanar graph with a chain $p = uv_1 \dots v_k w$, and let H be the subgraph of O obtained after deleting p. Let $\mathcal{R}_H$ be a proper* TTG *representation of H with two triangles $\triangle_u$ and $\triangle_w$ representing u and w, where $\triangle_u$ is a corner or a side triangle sharing an* X*-corner with $\triangle_w$. Then the representation $\mathcal{R}_O$ can be constructed from $\mathcal{R}_H$ by replacing the corner (or side) triangle $\triangle_u$ with a side (or a point) triangle $\triangle'_u$ and k side triangles.*

Proof. Let $\triangle_u = \triangle abc$ and $\triangle_w = \triangle cef$ with the common X-corner c. Since $\triangle abc$ is either a side or corner triangle, it has at least one side along the boundary of $\mathcal{R}_H$ and this side is incident to c. Assume then that this side is ac. Then the common boundary of $\triangle abc$ and $\triangle cef$ is contained in the side bc of $\triangle abc$. Assume without loss of generality that this common boundary is contained in the side ce of $\triangle cef$, thus making b, c and e co-linear. Fig. 3(a) gives one such possibility for $\mathcal{R}_H$. Then $\mathcal{R}_G$ can be obtained by first dividing the side or corner triangle $\triangle abc$ into the point or side triangle $\triangle abd$, respectively, and the side triangle $\triangle adc$ and finally if $k > 1$, then further dividing $\triangle adc$ into a set of side triangles; see Fig. 3(b)–(c). □

We would like to compute a peeling order f of O that will allow us to compute a sequence of proper TTG representations $\mathcal{R}_{O_1}, \dots, \mathcal{R}_{O_k}$ for the subgraph-sequence $O_1, \dots, O_k$ of O induced by f. We begin with a representation $\mathcal{R}_{O_1}$ of O_1 and repeatedly apply Lemma 7, leading to a final proper TTG representation $\mathcal{R}_O = \mathcal{R}_{O_k}$ of $O = O_k$. Note that each time a new chain is added, a corner or side triangle becomes a side or a point triangle in the new representation (from the proof of Lemma 7). Hence, each chain being added requires that one of the two endpoints of the chain is represented by a corner or a side triangle in the current

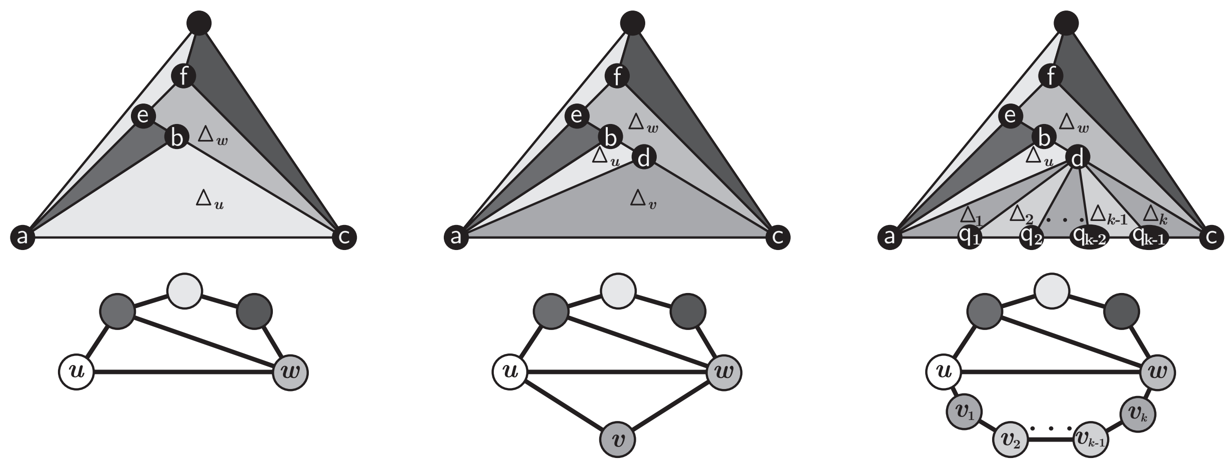


Fig. 3: Augmenting $\mathcal{R}_H$ in (a) to obtain $\mathcal{R}_G$ for $k = 1$ in (b) and for $k > 1$ in (c).

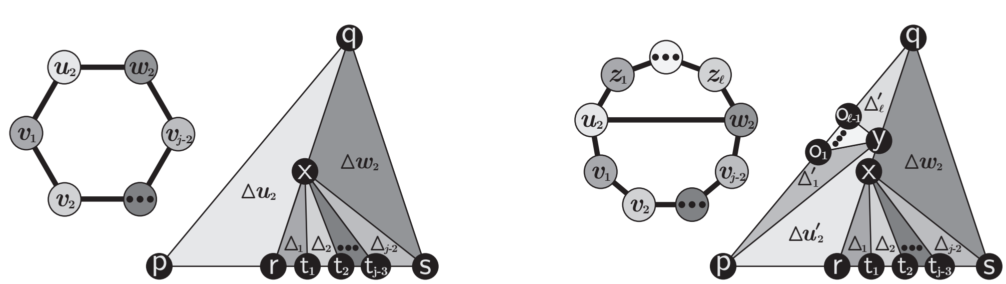
representation. Furthermore, point triangles remain unchanged in subsequent representations. This gives a one-to-one mapping, defined next, between the chains (or equivalently, internal faces) and their associated end-points in the peeling order.

Definition 8. *An* assigned peeling order *for a biconnected outerplanar graph $O(V, E)$ is a peeling order f together with an injection $\nu : \{f(3), \ldots, f(k)\} \xrightarrow{1-1} V$ where ν always assigns a face f of O to one of the endpoints of the chain that forms f in the peeling order.*

While every BOPG has a peeling order f, it may not be *assignable*. However, given an assigned peeling order for a BOPG, obtaining its proper TTG representation is a straight-forward exercise of applying Lemma 7.

Theorem 9. *A biconnected outerplanar graph with an assigned peeling order is a proper touching triangle graph.*

Proof. Let O be a BOPG with k internal faces having an assigned peeling order f and ν. Let $O_1, \ldots, O_k$ be the subgraph-sequence induced by f. We show by induction that O_j has a proper TTG representation for $j \in \{1, \ldots, k\}$. If $j = 1$, then $O_1 = f(1)$ and it has a proper TTG representation $\mathcal{R}(O_1)$; see Fig. 4(a). We then obtain $\mathcal{R}_{O_2}$ from $\mathcal{R}_{O_1}$ by Lemma 7, where w.l.o.g. we assume that the corner triangle of $\mathcal{R}_{O_1}$ is split into side-triangles. Note then that $\mathcal{R}(O_2)$ has no point triangles; see Fig. 4(b).



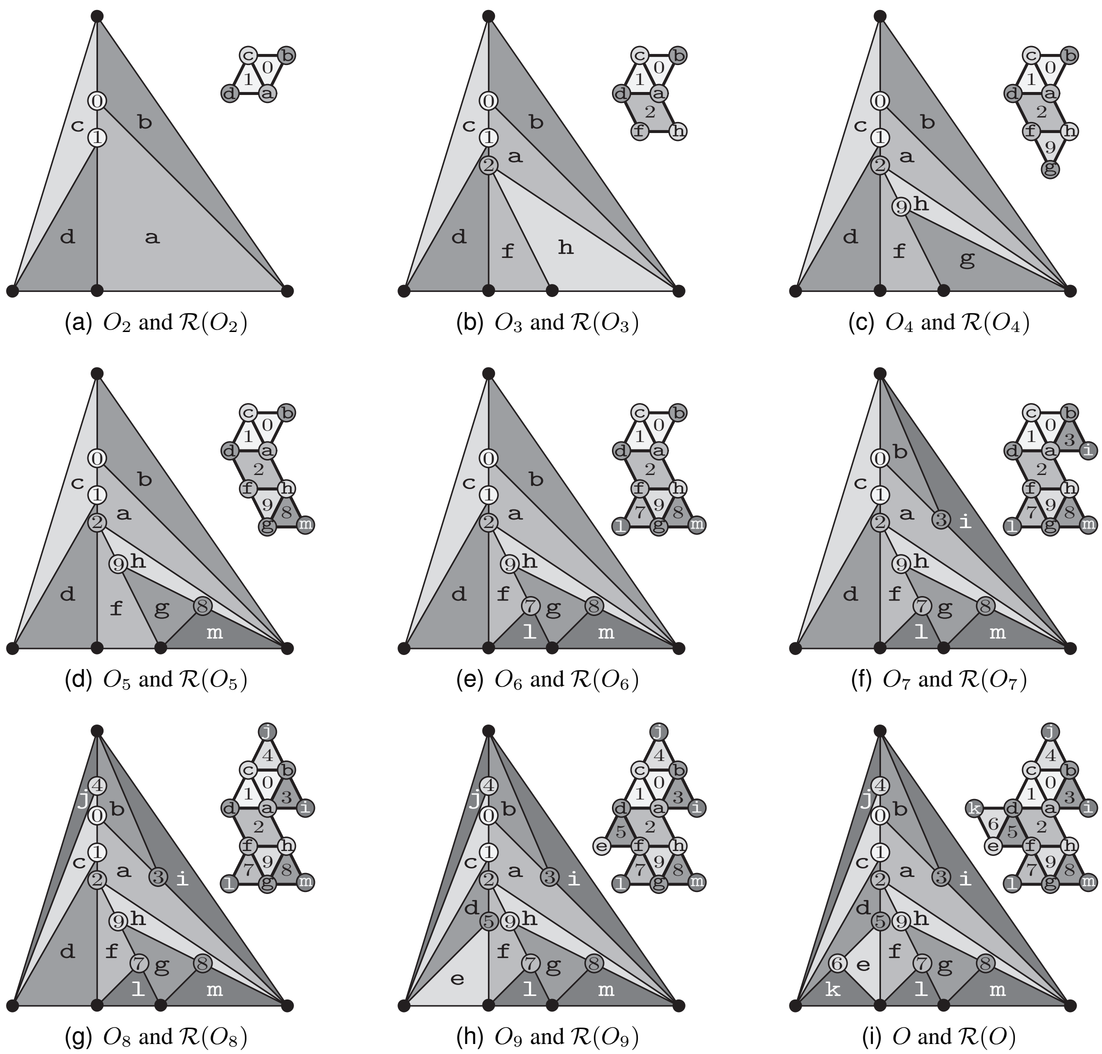
(a) O_1 and $\mathcal{R}(O_1)$ (b) O_2 and $\mathcal{R}(O_2)$

Fig. 4: First two steps in computing a proper TTG representation of a BOPG.

Assume then for $i \in \{2, \ldots, k-1\}$, there is a proper TTG representation $\mathcal{R}(O_i)$. Then $\mathcal{R}(O_{i+1})$ can be constructed from $\mathcal{R}(O_i)$ by applying Lemma 7. We show that the conditions of Lemma 7 hold. Let V_i denote the vertex set of O_i for $i \in \{1, \ldots, k\}$. Let P_i denote the set of vertices that have been assigned by ν previously, whereas S_i would denote the set of the remaining unassigned vertices. We maintain the invariant that P_i and S_i represent the point and side triangles, respectively in each $\mathcal{R}_{O_i}$. We argue that the conditions in Lemma 7 hold for each $\mathcal{R}_{O_i}$, $2 \leq i \leq k$. This is clearly true for $i = 2$.

Let $u_{i+1}, \ldots, w_{i+1}$ be the chain forming the face $f(i+1)$ where ν assigns $f(i+1)$ to u_{i+1}. Both endpoints are in O_i where $\triangle u_{i+1}$ and $\triangle w_{i+1}$ are their representing triangles. Since ν is an injection, then u_{i+1} has not been assigned by ν. Thus $u_{i+1} \notin \mathrm{P}_i$, and hence, $u_{i+1} \in \mathrm{S}_i$, i.e., $\triangle u_{i+1}$ is a side triangle by the induction invariant. Furthermore by Observation 2, the outeredge (u_{i+1}, w_{i+1}) of O_j, corresponds to some X-corner x common in both the triangles $\triangle u_{j+1}$ and $\triangle w_{j+1}$ in $\mathcal{R}(O_j)$. Thus all the conditions of Lemma 7 are met. Since the construction in Lemma 7 only creates the one point triangle for $\nu(j)$, the invariant is also maintained. Thus by induction, a proper TTG representation $\mathcal{R}(O_k)$ exists for $O = O_k$. $\square$

The proof of Theorem 9 gives an algorithm to construct a proper TTG representation of a BOPG given an assigned peeling order. Fig. 5 illustrates the construction of such a proper TTG representation.



(a) O_2 and $\mathcal{R}(O_2)$ (b) O_3 and $\mathcal{R}(O_3)$ (c) O_4 and $\mathcal{R}(O_4)$

(d) O_5 and $\mathcal{R}(O_5)$ (e) O_6 and $\mathcal{R}(O_6)$ (f) O_7 and $\mathcal{R}(O_7)$

(g) O_8 and $\mathcal{R}(O_8)$ (h) O_9 and $\mathcal{R}(O_9)$ (i) O and $\mathcal{R}(O)$

Fig. 5: Example proper TTG construction sequence for a biconnected outerplanar graph O.

4 Necessary Conditions for Proper TTG Representation

In this section we begin with a necessary conditions for proper TTG representation of *chord-connected* outerplanar graphs, (which are BOPGs with the stronger property that all chords form a connected subgraph). We then use this result to give necessary conditions for proper TTG representation of biconnected outerplanar graphs by means of a "chord-connected decomposition".

Definition 10. *The edges of a biconnected outerplanar graph $O(V, E)$ are of two types: the outerface f_o and the* chord-induced subgraph $chord(O)$ *of* O. *If $chord(O)$ is connected, then O is* chord-connected *(CC). The* chord-connected decomposition $decomp(O)$ *of O is the decomposition of O into chord-connected subgraphs $H_1, \ldots, H_k$ of O, where (i) $O = \bigcup H_i$, (ii) $chord(H_i)$ corresponds to a connected-component of $chord(O)$, and (iii) H_i contains* both *faces incident to each chord of $chord(H_i)$ in O.* Joining faces $join(O)$ *are the faces common to two or more such chord-connected subgraphs in $decomp(O)$.*

Fig. 6 illustrates the chord-connected decomposition of a BOPG O. Let k be the number of chord-connected subgraphs in the chord-connected decomposition of O. Then note that the weak dual T of O can be partitioned into k subtrees, $T_1, \ldots, T_k$, where each subtree T_j, for $j \in \{1, \ldots, k\}$, is the weak dual of H_j. The common intersection for a pair of chord-connected subgraphs H_i and H_j forms a joining face.

Having chord-connected subgraphs allows us to characterize proper TTG realizability in terms of chord-only faces. In particular, realizable chord-connected outerplanar graphs are fairly restricted as the next lemma show.

Lemma 11. *Let O be a chord-connected outerplanar graph with a proper* TTG *representation. Then there are at most two chord-only faces in O.*

Proof. Let $\mathcal{R}_O$ be a proper TTG representation of O. Then by Lemma 4, $\mathcal{R}_O$ induces a total charge function $\Pi_O : F' \to V$, where $F' \subset F_I = \{f_1, \ldots, f_k\}$ is the set of internal faces of O. Thus by the definition of a total charge function, $|F_I| \geq \sum_{v \in V} ch(O, v)) = ch(O)$. We now show by a counting argument that this implies at most two chord-only faces in O.

Take an arbitrary chord (u, w) of O. Let f_p and f_q be the two internal faces incident to (u, w). The chord (u, w) partitions O into two disjoint subgraphs, say P and Q, where $f_p \in P$ and $f_q \in Q$. Augment P with the face f_q to obtain P' and similarly augment Q with f_p to obtain Q'. Clearly, $P' \cap Q' = f_p \cup f_q$ and $P' \cup Q' = O$. We now claim that $ch(O) = ch(P') + ch(Q')$. Consider the charge of u. Since u has three common edges in P' and Q', it has $deg_{P'}(u) - 3$ edges in $P' \setminus Q'$ and $deg_{Q'}(u) - 3$ edges in $Q' \setminus P'$. Therefore, $deg_O(u) = deg_{P' \cap Q'}(u) + deg_{P' \setminus Q'}(u) + deg_{Q' \setminus P'}(u)$; which implies that $ch(O, u) = ch(P', u) + ch(Q', u)$. Similarly, $ch(O, w) = ch(P', w) + ch(Q', w)$. Since the only vertices common to P' and Q' are the vertices on the faces f_p and f_q and among these vertices only u and w has degree at least three in both P' and Q', this implies that $ch(O) = ch(P) + ch(Q)$.

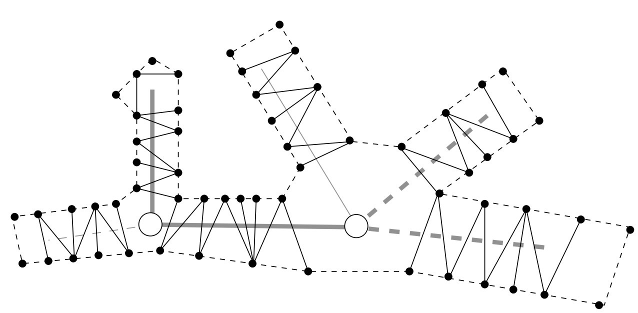


Fig. 6: Chord-connected decomposition of a biconnected outerplanar graph.

For any face $f_i \in F_I$, we can then define O_i to be the subgraph of O induced by the face f_i and the $deg(f_i)$ faces adjacent to f_i in O. Then $\mathcal{O} = \{O_i : deg(f_i) > 1\}$ forms a sufficient decomposition of O such that for any pair of adjacent faces f_p and f_q in $\mathcal{O}$, $O_p \cap O_q = f_p \cup f_q$ while $\bigcup_{O_i \in \mathcal{O}} O_i = O$. Recursively applying the relationship in the previous paragraph, we see that $ch(O) = \sum_{v \in O} ch(O, v) = \sum_{O_i \in \mathcal{O}} ch(O_i) = \sum_{O_i \in \mathcal{O}} \sum_{v \in O_i} ch(O_i, v) = \sum_{\substack{f \in F_I \\ deg_O(f) > 1}} \sum_{v \in f} ch(f, v)$.

For any face $f_i \in F_I$, the chords in f_i must be connected due to O being chord-connected. Hence, the $deg_O(f_i)$ chords in f_i must form either a cycle of length $deg_O(f_i)$ or a path of length $deg_O(f_i) - 1$, depending on whether f_i is a chord-only face or not. In case f_i is not a chord-only face, the path formed by the chords will have $deg_O(f_i) - 1$ internal vertices on the path each with degree 4 in O and two end-vertices on the path, each with degree 3 in O_i. On the other hand, if f_i is a chord-only face, then the cycle formed by the chords in f_i has $deg_O(f_i)$ vertices, each with degree 4 in O_i. Since all other vertices in O_i have degree 2 and since only the degree-4 vertices of O_i contribute one charge each, this gives either $ch(O_i) = deg_O(f_i)$ or $ch(O_i) = deg_O(f_i) - 1$ depending on whether f_i is a chord-only face or not. Thus if k is the number of chord-only faces in O, then $ch(O) = \sum_{f \in F_I}(deg_O(f) - 1) + k$. Furthermore since the weak-dual of O is a tree where each vertex represents an internal face of O and each edge represents a chord of O, we have $\sum_{f \in F_I} deg_O(f) = 2|C| = 2|F_I| - 2$. Since Π_O is a total charge function, this implies that $|F_I| \geq ch(O) = 2|F_I| - 2 + k$, which gives $k \leq 2$. Thus O can have at most two chord-only faces. □

Corollary 12. *Let O be a chord-connected outerplanar graph with a total charge function Π_O. Then Π_O can assign empty charge to at most $2 - k$ faces, where k is the number of chord-only faces in O.*

The corollary can be proved by an argument similar to that of Lemma 11. Since a CC subgraph of a biconnected outerplanar graph O is a face-induced subgraph of O, Lemma 11 together with Corollary 6 gives the following corollary.

Corollary 13. *A biconnected outerplanar graph O is a proper* TTG only if *each chord-connected subgraph of O has at most two chord-only faces.*

The following two lemmas imply that the total charge as well as an existence of a total charge function in a BOPG can be obtained using the chord-connected decomposition.

Lemma 14. *Let O be a biconnected outerplanar graph with chord-connected decomposition $decomp(O) = \{H_1, \ldots, H_k\}$. Then the total charge of O is the sum of the total charges of each H_i in $decomp(O)$, i.e., $ch(O) = \sum_{H \in decomp(O)} ch(H)$.*

Proof. We prove this by induction on k. Clearly, the claim holds for $k = 1$ when O is chord-connected. Suppose the claim holds for $k - 1$. Take the chord-connected subgraph $H_1 \in decomp(O)$ and define $O' = \bigcup_{j=2}^{k} H_j$. Thus $decomp(O) = decomp(O') \cup \{H_1\}$. Let f be the joining face shared by H_1 and O'. The only vertices in common between H_1 and O' are in f. Furthermore, each vertex of f has degree 2 either in H_1 or O'. Hence for each vertex v on f, either $ch(H_1, v) = 0$ or $ch(O', v) = 0$. Thus $ch(O) = ch(O') + ch(H_1)$. By induction hypothesis, $ch(O') = \sum_{j=2}^{k} ch(H_j)$. Therefore $ch(O) = \sum_{j=1}^{k} ch(H_j)$. □

Lemma 15. *Let O be a biconnected outerplanar graph with k biconnected subgraphs $H_1, \ldots, H_k$ such that all the faces of each subgraph are distinct except for one common face f of all the subgraphs. Then O has a total charge function Π_O* if and only if *each H_i has a total charge function Π_{H_i} such that Π_{H_i} allots a charge to f in at most one subgraph H_i.*

Proof. First suppose without loss of generality that each chord-connected subgraph H_i has a total charge function Π_{H_i} such that $\Pi_{H_i}(f) = \varnothing$ if $i > 1$. This induces a total charge function Π_O of O, where for each face $f' \neq f$ in H_i for some $i \in \{1, \ldots, k\}$, f' is assigned according ot Π_{H_i} and f is assigned according to

Π_{H_1}. Conversely, if O has a total charge function Π_O, then it induces a total charge function Π_{H_i} to each chord-connected subgraph H_i, obtained from Π_O restricted to the faces of H_i. Since f can be assigned to at most one H_i, at most one of these total charge function allots a chrge to f. □

We now have the following definition that gives our last necessary condition for a biconnected outerplanar graph to be a proper TTG.

Definition 16. *Let O be a biconnected outerplanar graph with chord-connected decomposition $decomp(O)$ and joining faces $join(O)$. Then O has a* satisfiable joining $\rho : join(O) \to decomp(O)$ *if each subgraph $H \in decomp(O)$ has zero, one, or two chord-only faces and at most two, one, or zero joining faces not assigned to H by ρ, respectively.*

Lemma 17. *A biconnected outerplanar graph O has a total charge function* only if *O has a satisfiable joining between its joining faces and chord-connected decomposition.*

Proof. Assume O has a total charge function Π_O. Let $decomp(O)$ and $join(O)$ be the chord-connected decomposition and joining faces of O. Apply Lemma 15 repeatedly for each of the joining faces in $join(O)$. Observe that any joining face f can give a charge for at most one chord-connected subgraph. This gives a function $\rho : join(O) \to decomp(O)$, where ρ maps each joining face to the chord-connected component for which it provides the charge. By Corollary 12, for any chord-connected component H with zero, one or two chord-only faces, the total charge function Π_H on H induced by Π_O leaves at most two, one or zero faces unchrged, respectively. Since every joining face in H unassigned to H by ρ must be uncharged by Π_H, ρ gives a satisfiable joining. □

A satisfiable joining of a BOPG $O(V, E)$ with k maximal chord-connected subgraphs can be found in $O(|V| + k^3)$ time by solving a maximum flow problem on a graph. See [2] for detailed proof. Summarizing our results, we have the following theorem.

Theorem 18. *A biconnected outerplanar graph O has a proper* TTG *representation only if it has a satisfiable joining.*

5 A Sufficient Condition for a Proper TTG Representation

While having a total charge function is a necessary condition for a planar graph to be a proper TTG, it is not a sufficient one. In this section, we describe a sufficient condition for a biconnected outerplanar graph to have a proper TTG representation. We first introduce the notion of a "peeling-compatible" total charge function.

Definition 19. *A total charge function Π_O of a biconnected outerplanar graph is* peeling-compatible *for a peeling order f if whenever the chains corresponding to two faces being added by f share a common end-vertex v, one of the two faces are assigned by to v Π_O.*

We now have the following lemma that together with Theorem 9 gives a sufficient condition for a biconnected outerplanar graph to be a proper TTG.

Lemma 20. *A biconnected outerplanar graph with a peeling order and peeling compatible total charge function has a peeling assignment.*

Proof. Let O be a biconnected outerplanar graph with a peeling order f on k internal faces. We first converts f and a peeling compatible total charge function Π_O into a peeling assignment ν. Let $O_1, \ldots, O_k$ be the sequence of subgraphs of O induced by f. For each step i, let $u_i, \ldots, v_i$ be the chain that forms the new face

$f(i)$. Then Π_O can assign $f(i)$ to either u_i or v_i or neither endpoint. If Π_O did not assign $f(i)$ to u_i, then ν assigns $f(i)$ to u_i. Otherwise ν assigns $f(i)$ to v_i.

We now show that ν gives a valid peeling assignment. Assume for a contradiction that ν is not a valid. Assume w.l.o.g. that ν assigns $f(i)$ to u_i (possibly by renaming) for each step i. Then if ν is not a valid peeling assignment, it cannot be an injection by Definition 19. Hence, there must exist some a pair of distinct faces $f(p)$ and $f(q)$ that have been assigned to the same vertex by ν. Thus, since ν has assigned $f(p)$ to u_p and $f(q)$ also to $u_q = u_p$, then Π_O has assigned neither $f(p)$, nor $f(q)$, to $u_p = u_q$. Thus the chains for the two faces $f(p)$ and $f(q)$ share the common end vertex $u_p = u_q$, but neither of them is assigned to it, contradicting the peeling-compatibility of Π_O. □

Given the algorithm in Lemma 20, we want to find a class of BOPG for which a peeling-compatible total charge function can be computed. We first have the following lemma that shows that the necessary condition for chord-connected outerplanar graphs in Lemma 11 is also sufficient.

Lemma 21. *Let O be a chord-connected outerplanar graph with at most two chord-only faces. Then O has a total charge function Π_O* where

(i) Π_O keeps two, one, or zero specified faces uncharged, when O has zero, one, or two chord-only faces, respectively;
(ii) chord-only faces are always charged by Π_O;
(iii) every vertex v in O is allotted exactly $ch(O, v)$ charges; and
(iv) Π_O is peeling-compatible with a peeling order.

Proof. We prove this lemma by giving an algorithm for computing a desired charge function Π_O. While we construct Π_O, we will find a peeling order f of O such that Π_O is peeling compatible with f. Note that a peeling order of O is nothing but an ordering of the internal faces of O. Let f_x and f_y be two special internal faces of O, where depending on the number of chor-only faces in O, these two faces can either be both chord-only faces, or both specified uncharged faces or one chord-only face and one uncharged face.

The algorithm starts by first constructing the minimal chord-connected subgraph H containing the two faces f_x and f_y. This is done by first finding a shortest path p in the chord-induced subgraph $chord(O)$ of O from a vertex v_x on f_x to a vertex v_y on f_y. Then H is the subgraph of O induced by all the faces incident to each internal vertex of p along with f_x and f_y. Thus each chord in the chord-induced subgraph of H, is either on p or has at least one end-point in p. We now claim that if such a chord is not on p, then it has exactly one end-point on p. Indeed, if a chord has both end-points on p, that would contradict p being the shortest path. We now show how the algorithm gives the peeling order on H and how Π_O assignes the faces in H. In particular, Π_O will not assign f_x and f_y to any vertex of H.

The peeling order starts by making f_x the first face and f_y the last face of H. The other faces of H are ordered as follows. Let $p = v_x, v_1, \ldots, v_l, v_y$. Then each face incident to v_i is order before any face incident to v_{i+1} for $1 \leq i < l$. Among the faces incident to a vertex v_i, we order them such that each chain creating a face must have endpoints that have already been added. Let $f_1 = f_x, f_2, \ldots, f_y$ gives this peeling order. Also suppose O_i be the subgraph induced by the first i faces in this list. Adjoining each new face f_i to form a new chord with end-vertices u_i and v_i has the effect of increasing the degree of both u_i and v_i. However in O_2 the degree of u_2 and v_2 is 3 and each other vertex has degree 2; resulting in no charge. We then maintain the invariant that for $i > 3$, when we are adding f_i, the face f_{i-1} is still unchrged. Consider now the case when we are adding f_i. If (u_i, v_i) is on p, then according to the vertex ordering along p, one of them has degree 2 in O_{i-1}. Otherwise, exactly one of the two vertex is on p. Then again the vertex not on p has degree 2. Thus in both cases, at most one of u_i and v_i has degree > 3 in O_i, hence at most one extra charge has been induced by the vertex common to f_{i-1} and f_i. We assign f_{i-1} for that charge. In this way, when we finish, we have assiged faces for each charge and f_x, f_y is still unchrged.

Next we process the faces adjacent to a chord-only face. Suppose f_x is a chord-only face. We add the faces adjacent to f_x in such an order that the first face is incident to v_x and each subsequent face being added is adjacent to face added immediately before. This ensures that except for the last face, the chain for each face have one endpoint with degree 2. Thus only one extra charge is induced and the newly created face is assigned for this charge. For the last face added, there are two extra charge induced and the newly created face as well f_x is then assigned for these two charges. Thus each chord-only face is charged by Π_O while each specified uncharged face remains uncharged.

Finally, all the remaining faces are added in arbitrary order, provided that each face f_i being added creates a new chord (u, v) in G_i that is incident to some previous chord. Such a choice for selecting f_i must always be possible given that since O is chord-connected. This order of adding faces ensures that for each face thus added, the corresponding chain has at least one end-point of degree 2 and hence at most one extra charge is induced. This newly created face is then assigned for this charge.

At each step the above algorithm assigns exactly as many faces to a vertex as is the amount of extra charge. Thus each vertex will be assigned exactly $ch(O, v)$ faces. This immediately implies that Tpi_O is a total charge function. Furthermore each chord-only face is charged by Π_O while each specified uncharged face remains uncharged. Finally whenever a face is added with a common endpoint of the corresponding chain with a previously added adjacent face, the common endpoint has degree > 3 (hence inducing an extra charge) either the newly created face or the adjacent face is assigned to that vertex for the extra charge. Therefore by definition Π_O is peeling compatible with the peeling order generated by the algorithm. □

Together Lemma 20, 21 and Theorem 9 imply that a chord-connected outerplanar graph with at most two chord-only faces is a proper TTG. This together with Lemma 11 gives the following theorem which *fully characterizes* when chord-connected outerplanar graphs are proper TTG.

Theorem 22. *A chord-connected outerplanar graph is a proper* TTG if and only if *it has at most two chord-only faces.*

However, there exist total charge functions that do not correspond to any assigned peeling order.

Lemma 23. *There exists a biconnected outerplanar graph with a total charge function for which there is no peeling order with a valid peeling assignment.*

Proof. Consider graph O in Fig. 7. It has two faces f_1 and f_2 with a common incident edge (u, v). The chord (u, v) splits O into two chord-connected subgraphs H_1 and H_2, containing f_1 and f_2, respectively. Both H_1 and H_2 have two chord-only faces and hence each has a total charge function Π_{H_1} and Π_{H_2}, by Lemma 21. Then by Lemma 14, $ch(O) = ch(H_1) + ch(H_2)$ and these two total charge function combined would give a total charge function Π_O for O. However, we now show that there is no assigned peeling order for O. A peeling order would start with a face from H_1 or H_2 and then proceed to H_2 or H_1, respectively. Assume

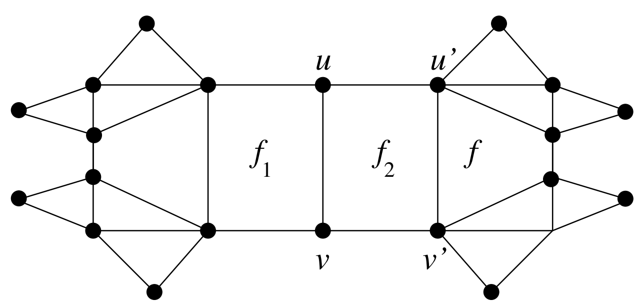


Fig. 7: Lemma 23 example.

without loss of generality that f_2 is added after f_1. Then f will be the face added to f_2. Then u' or v' must be assigned by ν. However, u' and v' are each incident to a chord-only face and adding the chord-only face and each of its incident faces requires to assign each face a charge. Since u' or v' are already assigned, there will be one face that cannot be assigned. □

We now use Lemma 21 to give a sufficient condition for a general biconnected outerplanar graph to be a proper TTG.

Theorem 24. *Let O be a biconnected outerplanar graph with chord-connected decomposition $decomp(O)$ = $H_1, \ldots, H_k$ where at most one H_i has two chord-only faces and all the remaining H_i $i \neq k$ have either one or zero chord-only faces. Then O has a proper* TTG *representation.*

Proof. Assume without loss of generality that H_1 has two chord-only faces. Applying Lemma 21 we can construct a peeling-compatible total charge function Π_{H_1} for H_1. Then we repeatedly apply Lemma 21 to compute a peeling-compatible total charge function Π_{H_i} for H_i with the additional restriction that for $i < j$, the joining face between H_i and H_j is uncharged in Π_{H_j}. Then the total charge function Π_{O_i} for $O_i = \bigcup_{i=1}^{k} H_i$ is peeling compatible where the order in which faces are added to construct O_i via repeated application of Lemma 21 gives a peeling order and the peeling compatibility of Π_{O_i} is a result of every Π_{H_i} being peeling compatible. Therefore for $O_k = O$, Π_{O_k} gives a peeling compatible total charge function for O, which can then give a peeling assignment by using Lemma 20. Once we get an assigned peeling order, we can use the algorithm in Theorem 9 to obtain a proper TTG representation. □

6 Conclusion and Open Problems

We gave a necessary condition and a slightly weaker sufficient condition for a biconnected outerplanar graph to have a proper TTG representation. Unfortunately we do not yet have a complete characterization because the sufficient condition is not necessary (and vice versa). For example, the graph in Fig. 8(a) does not satisfy the sufficient condition since it has more than one CC-subgraphs each with two full-chord faces. Yet it does have a proper TTG representation. We conjecture that the necessary condition is also not sufficient because the graph in Fig. 8(b) satisfies the necessary condition but likely does not have a proper TTG representation.

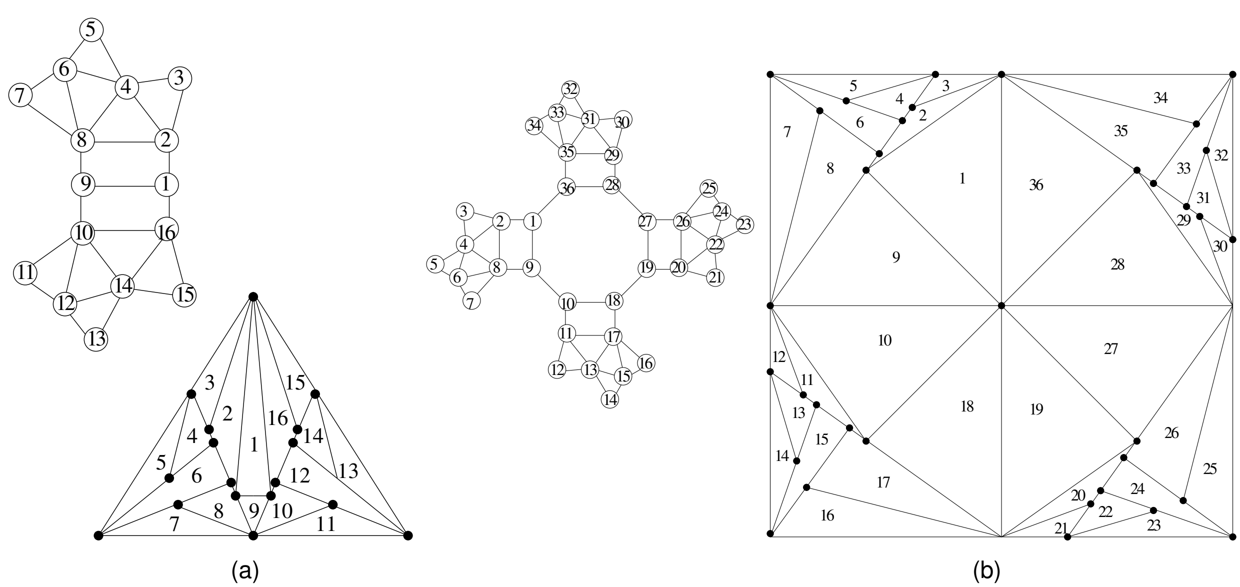


Fig. 8: (a) A graph and its proper TTG representations, (b) Another graph with a 4-sided TTG representation that we conjecture does not also have a proper TTG representation.

Thus the complete characterization for biconnected outerplanar graphs is still open. Naturally, the bigger problems of recognizing and characterizing the class of planar graphs with proper TTG representation are also open.

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